for radiation arising from transitions from the first resonance level at total pressures above 10^{-1} Torr and 10^{-2} Torr and spacings of 0.25 mm and 5.0 mm respectively, it is invalid for transitions from higher resonance levels over a wide range of pressures. This optically grey condition should be considered in a realistic treatment.

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Turbulent Diffusion * I.

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The problem of turbulent diffusion is re-examined. Its close connection with Schwinger's formulation of quantum field theory is established. For the transition probabilities we find a Dyson equation by using Zwanzig formalism with renormalization. The first nontrivial approximation of the memory kernel yields Saffman's result.

§ 1. The Problem

The phenomenon of turbulence was discovered by Reynolds using ink as an indicator. For a given turbulent field one may therefore consider the corresponding spread out of the ink, i. e. we require the transition probability.

If the test particle is very small, one may expect it to follow immediately the motion of the surrounding fluid, according to [1]

$$\dot{x} = v(t, x). \tag{1}$$

For a finite size of the particle, we expect some kind of relaxation which has been carefully estimated by Hinze [2]. If v were independent of x, we should have Brownian motion. This may serve us later on as a reference [3].

Starting from a fixed point, say $x = x_0$, $x(t)$ becomes a stochastic process in the course of time and hence the right-hand side of Eq. (1) develops into a stochastic function of another one, which is difficult to handle. In principle, the problem is a follows: Let the statistics of $v(t, x)$ be given by the sequence of its joint-probability distributions [4]:

$$\begin{aligned} &P_1(t_1, x_1 : v_1)\,\mathrm{d}v_1 \\ &P_2(t_1, x_1 : v_1; t_2, x_2 : v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2 \\ &\vdots \end{aligned} \tag{2}$$

For any realization ω of the turbulent fluid we have

$$x(x_0, t, \omega) = x_0 + \int_0^t v\,(t', x(t', \omega), \omega)\,\mathrm{d}t'. \tag{3}$$

For different realizations we get a distribution function $p(t, x_0 : x)$ of the positions at time t which we may approximate as follows: Let us assume that the paths are rather continuous and can be approximated by a polygon. Then for $t = n\,\tau$, we have:

$$\begin{aligned} p(t, x_0 : x) \approx \int \cdots \int P_{n+1}[0, x_0 : (x_1 - x_0)/\tau;& \\ \tau; x_1 : (x_2 - x_1)/\tau; \ldots; t, x_n : (x - x_{n-1})/\tau]& \\ \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n& \end{aligned} \tag{4}$$

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and expect the probability in question to be given in the limit $\tau \to 0$. In the following, however, we shall not try this directly.

§ 2. Schwinger's Equation

The above mentioned difficulty can be avoided if the problem is given a formulation in which t and x are considered as independent variables (as they are within v). This is achieved by the continuity equation:

$$\frac{\partial q(t,x)}{\partial t} + \frac{\partial}{\partial x}\big(v(t,x)\; q(t;x)\big) = 0 \qquad (5)$$

which will hold for (almost) any realization. With the initial condition

$$q(t=0,x) = \delta(x-x_0) \qquad (6)$$

its solution offers the very same information contained in Eq. (1). Furthermore, we get by Bayes' rule

$$p(t, x_0 : x) = \int q(t,x;\omega)\;\mathrm{Prob}(\omega)\;\mathrm{d}\omega = \mathcal{E}\{q(t,x)\} \qquad (7)$$

where $\mathcal{E}$ denotes the expectation with respect to the random velocity field.

The advance of the new formulation is that now $v(t,x)$ and $q(t,x)$ are treated on an equal footing as two interacting fields.

The one-way causal connection is expressed by the fact that the stochastics of one of the fields (v) is to be considered as a given one, and that of q is to be determined by the equation of motion (5).

Let us introduce a characteristic functional as a description of both stochastic fields [5]:

$$\Phi(\alpha,\beta) = \mathcal{E}[\exp\{i \iint \alpha(t,x)\; v(t,x)\;\mathrm{d}t\,\mathrm{d}x + i\iint \beta(t,x)\; q(t,x)\;\mathrm{d}t\,\mathrm{d}x\}] \qquad (8)$$

where α and β are test functions with compact supports. The above mentioned causality of our statistics is then expressed first of all by the boundary condition:

$$\Phi(\alpha, \beta=0) = \Phi_{\mathrm{M}}(\alpha) \qquad (9)$$

where $\Phi_{\mathrm{M}}(\alpha)$ denotes the stasictics of the medium, which is asumed to be known. Here and in the following it is assumed that it will not be disturbed by the presence of the test particle. (In the case of stochastic acceleration, however – as is of interest in plasma physics – a recoupling has to be taken into account [6].) On this assumption, it is a very natural generalization – at least a convenient one – to allow also continuous initial distributions $q_0(x)$. Of course, we shall assume $q_0(x)$ to be independent of the field v, and hence have

$$\Phi\big(\alpha, \gamma(x)\,\delta(t)\big) = \Phi_{\mathrm{M}}(\alpha)\,\exp\{i\int \gamma(x)\; q_0(x)\;\mathrm{d}x\}\,. \qquad (10)$$

Besides this we have, of course,

$$\Phi(0,0) = 1\,, \qquad (11\,\mathrm{a})$$

$$\Phi \text{ positive definite}^{5}. \qquad (11\,\mathrm{b})$$

But the dynamic restriction is imposed by the equations of motion. Our attention is focused on

$$p(t,x) = (\delta\Phi/\delta\beta(t,x))_{\alpha=0,\,\beta=0}\,. \qquad (12)$$

In order to fix its time dependence, let us consider

$$\partial_t\big(\delta\Phi/\delta\beta(t,x)\big) = i\,\mathcal{E}\,\frac{\partial q(t,x)}{\partial t} \times \exp\{i\iint(\alpha v + \beta q)\;dt\,\mathrm{d}x\}\,. \qquad (13)$$

Here ∂_t denotes derivation with respect to time and δ means variational derivation (for a definition suitable for our purposes [7], Appendix B).

By means of Eq. (5) we rewrite Eq. (13), using a shorthand notation for the variational derivative, in the form

$$\partial_t\,\delta_\beta\,\Phi = -i\,\partial_x\,\mathcal{E}\{v\,q\,e^{i\cdots}\} \qquad (14)$$

where the linear operations $\partial_x(=\partial/\partial x)$ and $\mathcal{E}$ have been interchanged. The right-hand side can be rewritten as an expectation:

$$\partial_t\,\delta_\beta\,\Phi = i\,\partial_x\,\frac{\delta^2\Phi}{\delta\alpha(t,x)\;\delta\beta(t,x)}\,. \qquad (15)$$

Owing to its similarity with corresponding relations in quantum electrodynamics let us call this a Schwinger equation [8]. It expresses an initial value problem. Within the scope of Schwinger's theory it can be formulated asymptotically in terms of scattering amplitudes. We get for the finite time $t=0$:

$$\delta\Phi(\alpha,\beta)/\delta\beta(t=0,x) = \mathcal{E}\{i\,q_0(x)\;e^{i\cdots}\} = i\,q_0(x)\;\Phi(\alpha,\beta)\,. \qquad (16)$$

which shows a recoupling of Φ into itself by the initial conditions. Hence we are actually dealing with a nonlinear problem.

It is perhaps of interest that this nonlinearity can be moved from the initial conditions to the equation of motion by the transition to the cumulant functional Z defined by [7]

$$\Phi(\alpha,\beta) = \exp\{Z(\alpha,\beta)\}\,. \qquad (17)$$

We now get for the initial condition the very explicit expression:

$$(\delta Z/\delta\beta(t,x))_{t=0} = i\,q_0(x) \tag{18}$$

but Schwinger's equation now reads:

$$\partial_t \frac{\partial Z}{\delta\beta(t,x)} = i\,\partial_x \left\{ \frac{\delta^2 Z}{\delta\alpha(t,x)\,\delta\beta(t,x)} + \frac{\delta Z}{\delta\alpha(t,x)}\,\frac{\delta Z}{\delta\beta(t,x)} \right\} \tag{19}$$

However, for our present task (determination of the transition probability) the (original) formulation (15) is quite sufficient. The reason rests in some kind of a gauge invariance: If the test function β is changed into $\lambda\beta$ (λ a constant), the equation of motion (15) is not altered. Hence, any solution $\Phi(\alpha,\beta)$ of Eq. (15) immediately yields another one, $\Phi(\alpha,\lambda\beta)$. This fact merely reflects the homogeneity of Eq. (5), as is evident from the definition (8) of Φ! From this gauge invariance for Φ we conclude that monomials in β are solutions of Schwinger's equation, and we have no hierarchic coupling between different degrees if we use a Taylor expansion with respect to β. This holds particularly for the linear part

$$i\,\varphi(t,x\,|\,\alpha) = (\delta\Phi/\delta\beta(t,x))_{\beta=0} \tag{20}$$

for which we get

$$\partial_t\,\varphi(t,x\,|\,\alpha) = i\,\partial_x\,\delta_{\alpha(t,x)}\,\varphi\,. \tag{21}$$

It should be noted that now the initial condition does not further recouple to φ as compared with (16):

$$\varphi(t=0,x\,|\,\alpha) = q_0(x)\;\Phi_{\mathrm{M}}(\alpha)\,. \tag{22}$$

In principle, our task is to solve Eq. (21) with Eq. (22) and finaly set $\alpha=0$. Hence Schwinger's equation, despite its being only a necessary restriction for the characteristic functional, yields the complete answer to our problem. However, one may raise the question whether Eqs. (10), (11), (15) and (16) are also sufficient to determine the characteristic functional completely.

§ 3. The Dyson Equation

We are required to operate on the solution of Eq. (21) by

$$P := \text{“set } \alpha=0\text{”}. \tag{23}$$

According to Zwanzig [9], this can be done to some extent before solving Equation (21). This equation looks rather like a Schrödinger equation

$$\partial\varphi/\partial t = i\,\mathcal{H}(t)\;\varphi \tag{24}$$

together with some initial condition

$$\varphi(t=0) = \varphi_0\,. \tag{25}$$

In our case, the Hamiltonian

$$\mathcal{H}(t) = \partial_x\,\delta_{\alpha(t,x)} \tag{26}$$

is time dependent. This introduces some slight modifications. For convenience I shall set out the whole derivation in a shortened form.

In our case, P is a projector, but this is not essential. Let the "complementary" operator be denoted by

$$Q = 1 - P\,. \tag{27}$$

For convenience Zwanzig assumes that

$$Q\,\varphi_0 = 0 \tag{28}$$

for the initial conditions. (Otherwise the result will only be approximate.) In our case this will not hold since

$$Q\,\varphi_0 = q_0(\Phi_{\mathrm{M}}(\alpha) - 1) \tag{29}$$

but it can be achieved rigorously by a "renormalization" of the equation of motion: Introducing ψ by

$$\varphi(t,x\,|\,\alpha) = \psi(t,x\,|\,\alpha)\;\Phi_{\mathrm{M}}(\alpha) \tag{30}$$

then yields

$$\psi_0 = q_0 \tag{31}$$

and hence Eq. (28) is satisfied.

The equation of motion now reads

$$\partial_t\,\psi = i\,\partial_x\left(\delta_{\alpha(t,x)} + \delta\ln\Phi_{\mathrm{M}}/\delta\alpha(t,x)\right)\psi \tag{32}$$

according to the renormalized Hamiltonian

$$\mathcal{H}(t) = \partial_x\left(\delta_{\alpha(t,x)} + L(t,x\,|\,\alpha)\right) \tag{33}$$

from which the definition of L is obvious. The crucial point is that ψ may serve as well as φ in order to get $p(t,x)$ since by the definition of P in our case we have

$$P\,\psi = P\,\varphi\,. \tag{34}$$

[It should be borne in mind that $\Phi_{\mathrm{M}}(\alpha=0)=1$.]

A further assumption postulates that P be a Schrödinger-type operator that is independent of time:

$$\partial P/\partial t = 0\,. \tag{35}$$

This is obviously fulfilled by our definition. However, it may be convenient also to introduce a "Heisenberg" version of P by means of

$$P_{\mathrm{H}}(t)\;\psi_0 = P_{\mathrm{S}}\,\psi(t) \tag{36}$$

where we have added the index S for "Schrödinger".

Let us now act upon the equation of motion with P_S! Equation (35) allows P and ∂_t to be interchanged and we get

$$\partial_t P_S \varphi(t) = i P_S \mathcal{H}(t) \left(P_S \varphi(t) + Q_S \varphi(t)\right). \tag{37}$$

Similar properties as for P_S hold for Q_S, and so we get symmetrically

$$\partial_t Q_S \psi(t) = i Q_S \mathcal{H}(t) \left(P_S \psi(t) + Q_S \psi(t)\right). \tag{38}$$

The latter equation will be used in order to eliminate the unknown $Q_S \psi(t)$ from Equation (37). Equation (38) is rewritten as an integral equation by using the initial condition (28):

$$Q_S \psi(t) = i \int_0^t Q_S \mathcal{H}(t') \, P_H(t') \, dt' \, \psi_0 + i \int_0^t Q_S \mathcal{H}(t') \, Q_S \psi(t') \, dt' \tag{39}$$

or

$$= i \, \Omega(t) \left(P_H(t) \, \psi_0 + Q_S \psi(t)\right). \tag{40}$$

Here, Ω has to be defined as follows:

$$\Omega(t) \, f(t) = \lim_{\tau \to t} \int_0^t Q_S \mathcal{H}(t') \exp\{(t' - \tau) \, \partial_t\} \, dt' \, f(t). \tag{41}$$

Solving for $Q_S \psi(t)$, we may insert into Eq. (37) and obtain

$$dP_H(t)/dt = i P_S \mathcal{H}(t) \left[1 - i \, \Omega(t)\right]^{-1} P_H(t). \tag{42}$$

The corresponding initial condition is

$$P_H(t=0) = 1. \tag{43}$$

We have thus derived a kinetic equation for the Heisenberg operator $P_H(t)$ which yields the only relevant information. The "inverse bracket expression" in Eq. (42) is to be regarded as an expansion.

§ 4. Saffman's Equation

The Eqs. (42), (43) together with Eq. (12) afford a useful tool for further treatment. Hence we have explicitly for $p := P \psi$

$$\partial p/\partial t = i P_S \, \partial_x (\delta_{a(t,x)} + L(t, x \mid a)) \cdot [p(t,x) + i Q_S \int_0^t \partial_x (\delta_{a(t',x)} + L(t', x) \, a) dt' \, p(t', x) + \ldots]. \tag{44}$$

The first term in the expansion does not yield anything and we get as a first nontrivial approximation

$$\begin{aligned} \partial p/\partial t &= - \partial_x P_S \, \delta_{a(t,x)} \int_0^t \partial_x L(t', x \mid a) \, dt' \, p(t', x) \\ &= - \partial_x \lim_{\substack{y \to x \\ a \to 0}} \int_0^t \partial_x \frac{\delta^2 \Phi}{\delta a(t,x) \, \delta a(t', y)} \, dt' \, p(t', x) \\ &= a_x \lim_{y \to x} \partial_x \int_0^t \varrho(t, x, t', y) \, dt' \, p(t', x) \end{aligned} \tag{45}$$

where ϱ is the correlation function of the turbulent velocity field. In the case of homogeneous turbulence this simplifies to

$$\partial p/\partial t = \int_0^t \varrho(t, t', 0) \, dt' \, \partial_x^2 \, p(t', x) \tag{46}$$

since ϱ is a symmetric function and hence $\partial_x \varrho(x-y)$ vanishes for $x = y$. This above result was obtained by Saffman [10], in 1969, using Wiener-Hermite technique, and again by Phythian [11], in 1972, using variational techniques. As an approximation it corresponds to Bourret's 1-ficton approximation [12].

It is credible that for times large compared with the correlation time, we may consider $\varrho(t, t', 0)$ as a δ-function and hence obtain an instantaneous equation of the form [13]

$$\partial p/\partial t = \tfrac{1}{2} \langle v^2 \rangle \, \partial^2 p/\partial x^2 \tag{47}$$

which has been suggested by Taylor in the early twenties.

However, as has been emphasized by Saffman, the short-time behaviour is poorly described by the approximation (46). I do not want to repeat his argument but merely state that the special case where v does not depend on x, which can be treated completely at least in the case of a Gaussian-turbulent field (Φ_M a Gaussian functional), is not covered by Equation (46). Hence this may serve as an "unperturbed Hamiltonian". In subsequent papers, we shall follow these lines and also elaborate the three-dimensional case and the question of non-Gaussianity.

Added in proof: R. Kraichnan brought to my attention that Bourret suggested Saffman's result already in 1959 [14].

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MPD-Ringkanalströmungen in gekreuzten Fremdfeldern für willkürliche Hall-Parameter

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MPD Annular Duct Flow in a Crossed-Field Device for Arbitrary Values of the Hall Parameter

The steady, supersonic, barotropic, and inviscid flow of a rarefied gas plasma through an axisymmetric crossed-field device (imposed axial electric field and radial magnetic induction) is being analyzed, for arbitrary values of the Hall parameter. A previously developed perturbation method is used to obtain closed form solutions for the electric current density, flow velocity, and mass density within the annular duct. Conditions for optimal values of the Hall parameter are given.

1. Einleitung

In Fortsetzung einer früheren Arbeit [1] über Freistrahl- und Rohrströmungen eines verdünnten barotropen Gasplasmas im „gestreckten" Fremdmagnetfeld mit axialem elektrischem Feld bei schwachem Hall-Effekt wird im folgenden die durch *gekreuzte* Fremdfelder beeinflußte Ringkanalströmung für *beliebige* Werte des Elektronen-Hall-Parameters ψ theoretisch untersucht. Bei derartigen axialsymmetrischen Anordnungen bedient man sich i. allg. eines Topfkernmagneten zur Erzeugung der (im Idealfall) rein radialen fremdmagnetischen Induktion

$$\hat{\boldsymbol{B}}^{00} = \hat{B}_{\hat{r}}^{00}(\hat{r})\, \boldsymbol{e}_{\hat{r}}\ ^{*}.$$

Solche ringförmigen Hall-Beschleuniger dürften bei der Entwicklung von Raumfahrttriebwerken [2] (z. B. für Lageregelungssysteme) von Interesse sein.

Zwecks Straffung der Darstellung wird eine weitestmögliche Anlehnung an Verfahrensweisen und Bezeichnungen von [1] angestrebt. So wird unter der Annahme kleiner magnetischer Reynolds- und Druckzahlen (verglichen mit Eins) wieder die dort dargestellte Linearisierungsmethode für die dimensionslos gemachten Grundgleichungen der Magnetogasdynamik verwendet. Im übrigen sei für eine ausführlichere Darstellung auf den Bericht [3] verwiesen.

2. Problemstellung und Grundgleichungen

Es wird die stationäre und reibungsfreie Strömung eines barotropen Lichtbogenplasmas in der in Abb. 1 schematisch dargestellten ringförmigen und axialsymmetrischen Anordnung (Innenradius $\hat{r}_0$, Außenradius $\hat{R}$) untersucht. Die Mantelrohre seien nichtleitend und unmagnetisch. Zwischen den beiden im Abstand $\hat{L}$ befindlichen – angenähert etwa durch Drahtnetze realisierbaren – Äquipotentialflächen bestehe die Potentialdifferenz $\hat{V}$. Die Überschall-Einlaufgeschwindigkeit $\hat{\boldsymbol{v}}^{00}\{0, 0, \hat{v}_{\hat{z}}^{00}\}$, $\hat{v}_{\hat{z}}^{00} = \text{const}$, im Querschnitt $\hat{z} = 0$ wird vorgegeben, ebenso die Dichte $\hat{\varrho}^{00} = \text{const}$ dort. Gesucht sind elektrische Stromdichte $\hat{\boldsymbol{j}}$, Strömungsgeschwindigkeit $\hat{\boldsymbol{v}}$ und Massendichte $\hat{\varrho}$ des Plasmas im Ringraum

$$\hat{r}_0 \leqq \hat{r} \leqq \hat{R}.$$

Sämtliche Materialkoeffizienten werden konstant angenommen. Im verallgemeinerten Ohmschen Gesetz werden der Ionenschlupf sowie der Gradient des Elektronengaspartialdruckes vernachlässigt. Ausgangspunkt für die Untersuchungen bilden die in [1]

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* ^ kennzeichnet dimensionsbehaftete Größen.